Chapter 9.2: Conic Sections—Ellipses

What happened if you sat upon a flexible circle? You'd squish it down forming an oval, or what we formally call, an ellipse! While a circle has only one radius, you can think of an ellipse as being a squished circle with *two different radii*, a horizontal *x*-radius, r_x, and a vertical *y*-radius, r_y. Either one of these may larger or smaller than the other, depending on whether the ellipse is "squished" horizontally or vertically. The larger of these two radii is half the **major axis**, while the smaller of these two is half the **minor axis**. The major axis contains two points each called a **focus**, and together known as the **foci**.

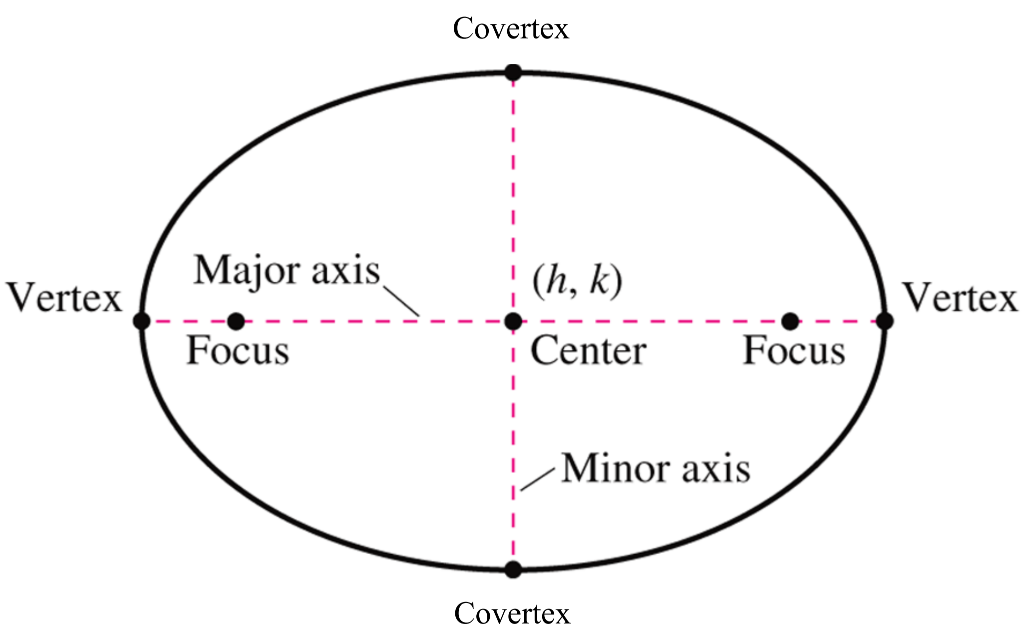
Geometrically, an ellipse occurs as the cross section when you slice a right circular cone at *an angle straight through from one side to the other*.

General Equation of an Ellipse: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

The General Equation of an Ellipse is given as

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,\ A \neq C \text{ and } AC > 0\ (A \text{ and } C \text{ are same sign})$$

While in a Circle, *A* & *C* are the same sign and magnitude, in an Ellipse, they are the same sign but *different* in magnitude! Think of the coefficients of the squared terms as being related to the *x*-radius, r_x, and *y*-radius, r_y. If they are the same, there is only one radius, and you have a circle. If they are the same sign but different, you have two different radii, and you have an ellipse!



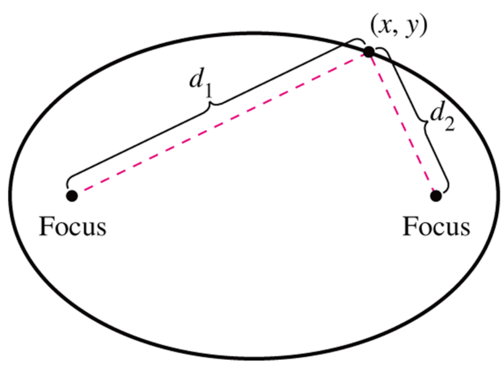
Like a circle, an ellipse has a center. In the diagram above, the center is still at (h, k). This ellipse is wider than it is tall, so the horizontal axis is longer than the vertical axis. For this reason, the horizontal axis is called the **major axis** and the shorter vertical axis is called the **minor axis**. The endpoints of the major axis are called the vertices (or each is a **vertex**). The endpoints of the minor axis are called the covertices (or each is a **covertex**). For this horizontal configuration, the distance from the center to a vertex is the *x*-radius, $a = r_x$. The distance from the center to a covertex is the *y*-radius, $b = r_y$ (these would switch for a vertical configuration.)

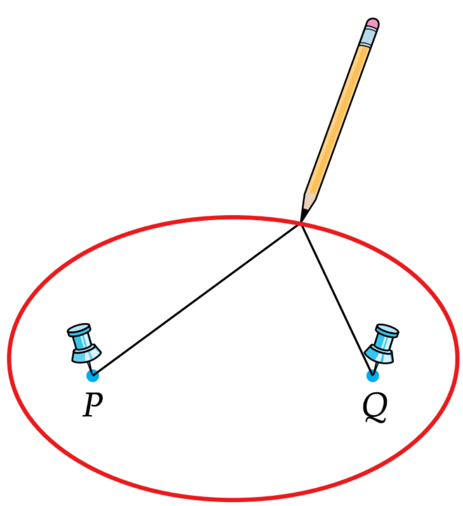
Again, the major axis contains two points called the **foci**. These foci have very important roles in an ellipse. The farther from the center the foci are, the flatter the ellipse. The closer they are, the rounder the ellipse. The distance from the center to a focus is called the focal length, c.

Locus Definition of an Ellipse

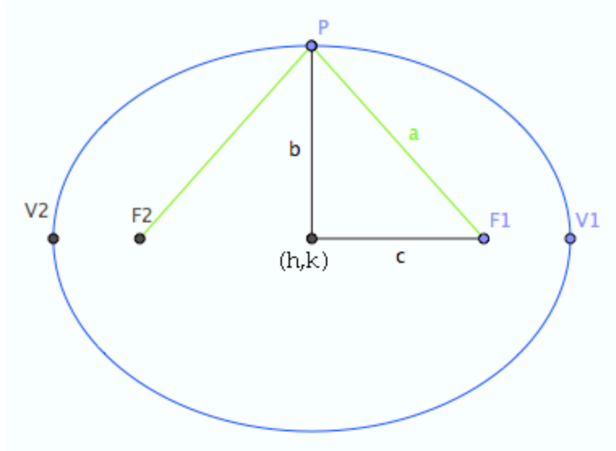
A circle is the set of all coplanar points such that the sum of the distances from two fixed points is constant. The fixed points are the foci and the constant sum is $2a$, the length of the major axis.

$$d_1 + d_2 = 2a$$





There is a special relationship among *a*, half the major axis, *b*, half the minor axis, and *c*, the focal length.



By the Pythagorean Theorem,

$$b^2 + c^2 = a^2$$

Rewriting this, we arrive at,

$$a^2 - b^2 = c^2$$

We can now discuss a way to measure the "roundness" of an ellipse like we did with the circle. This is called the **eccentricity**, e, and is defined as

$$e = \frac{c}{a}$$

For an ellipse, we know that $0 < e < 1$. The closer e is to zero, the rounder the ellipse. The closer e is to one, the flatter the ellipse.

Example 1:
Astronomy The former planet Pluto, while no longer a planet, still follows an elliptical orbit around the sun, which is at one focus of its orbit. The minimum and maximum distances of Pluto from the sun occur at the vertices of the ellipse. The minimum distance, at its perihelion, is 2.7 billion miles, and the maximum distance, at its aphelion, is 4.5 billion miles. What is the eccentricity of Pluto. *Pluto's eccentricity is the greatest of all the planets, while Venus, with an eccentricity of 0.0068, is the smallest. Earth's is 0.017.

Example 2:
Put the general equation of the ellipse given by $9x^2 + 16y^2 - 126x + 64y = 71$ into standard form. Sketch the ellipse, and identify all the parts including the domain, range, and eccentricity.

Standard Equation of an Ellipse

The standard form of the equation of an ellipse with center (h, k), major axis length of $2a$, and minor axis length of $2b$, where $a > b$, is

Horizontal Major Axis: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

or

Vertical Major Axis: $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$

The Foci lie on the Major Axis, c units from the center with $a^2 - b^2 = c^2$

Here's what it all looks like:

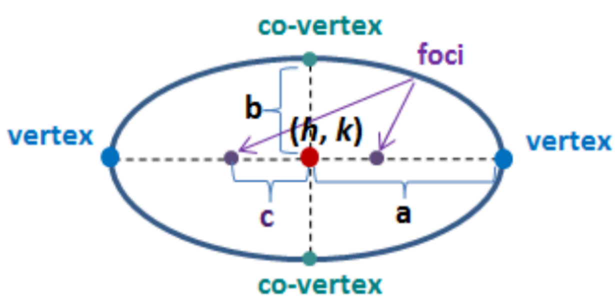
<table>
<tr><th>Horizontal Ellipse</th><th>Vertical Ellipse</th></tr>
<tr><td>

*center at (0,0): $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Standard Form: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

$a^2-b^2=c^2 \qquad e=\frac{c}{a}$

Center: (h,k) Foci: $(h\pm c,k)$
Vertices: $(h\pm a,k)$ Covertices: $(h,k\pm b)$



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*center at (0,0): $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

Standard Form: $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$

$a^2-b^2=c^2 \qquad e=\frac{c}{a}$

Center: (h,k) Foci: $(h,k\pm c)$
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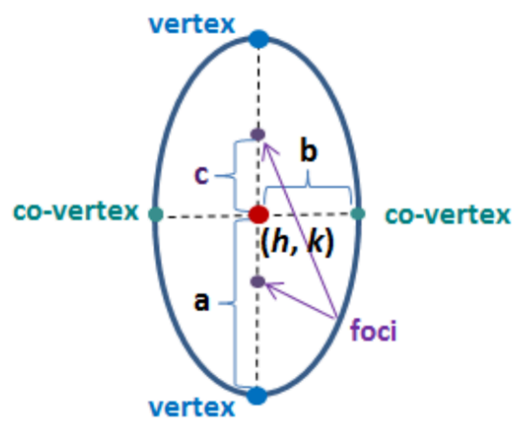

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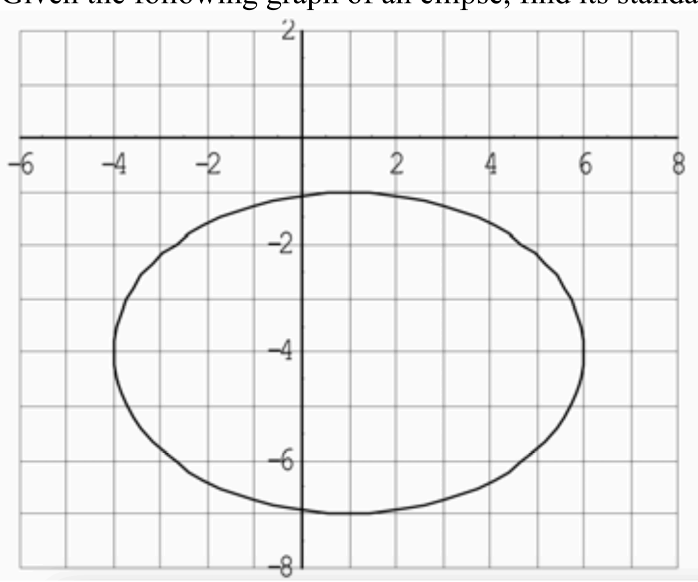
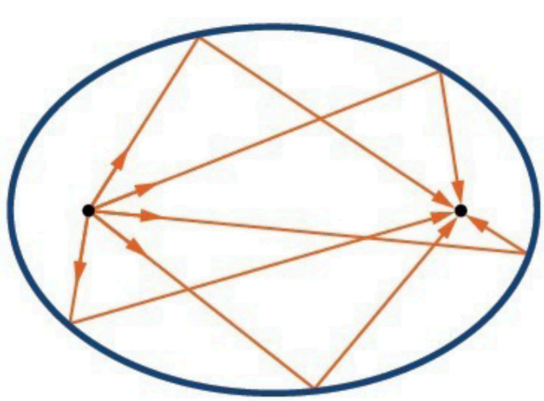
Example 3:
Given the following graph of an ellipse, find its standard equation, eccentricity, foci, and domain and range.

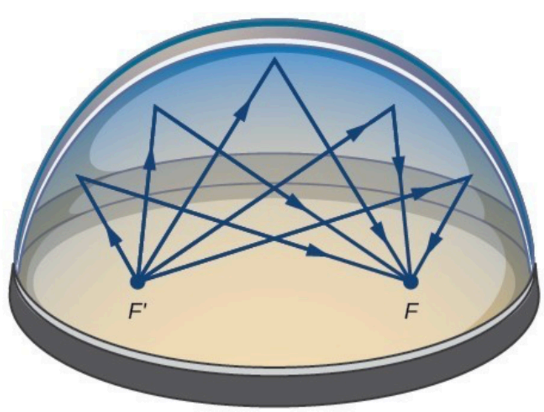


Example 4:
Given the equation of an ellipse, $25x^2 + 9y^2 = 225$, sketch the graph, then find its eccentricity, foci, and domain and range.

Example 5:
Find the standard equation of an ellipse whose vertices are $(0,2)$ & $(4,2)$, and whose eccentricity is $e = \frac{1}{2}$. Sketch the graph of the ellipse, find its foci, and domain and range.

An ellipse has a very useful reflective property. Anything emanating from one focus will always bounce of the ellipse and pass through the second focus. This happens in 2D, like in Lithotripsy, which is used to smash kidney stones, and in 3D, as in whispering rooms.





Example 6:
An elliptical pool table has only one hole on the surface. There are only two balls in the game, a cue ball and a target ball. The object of the game is to hit the target ball with the cue ball and deposit the target ball into the hole after one bounce off the elliptical cushion. The cue ball can be place anywhere on the table. Given the following dimensions of the table, describe a technique that will ensure that the target ball will go into the hole every time given proper placement of the cue ball.

The table is 5 ft long and 4 ft wide. The hole is placed 1.5 ft from the center of the table.