Chapter 9.3: Conic Sections—Hyperbolas

We're learning about someone new today, so be friendly, and say it with me:

"Hi, Perbola!"

Ok, speaking hyperbolically, that was the best joke of all time, right?

Anyway, to get a hyperbola geometrically, you need a double-knapped cone, like in the picture to the right. If you slice with a plane perpendicular to the base of the cone you get a hyperbola, which consist of two **branches**.

Algebraically, we can look at how the general equation of the hyperbola compares with those of the circle and ellipse.

General Equation of a Hyperbola: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

The General Equation of a Hyperbola is given as

$$Ax^2 - Cy^2 + Dx + Ey + F = 0 \text{ or } -Ax^2 + Cy^2 + Dx + Ey + F = 0, \text{ where } A > 0, C > 0$$

Essentially, a hyperbola algebraically can be identified by noticing the signs of the squared terms are **opposite in signs**, one positive and one negative. Depending on which one is positive or negative determines whether we have a vertical or horizontal hyperbola (more on that in a bit).

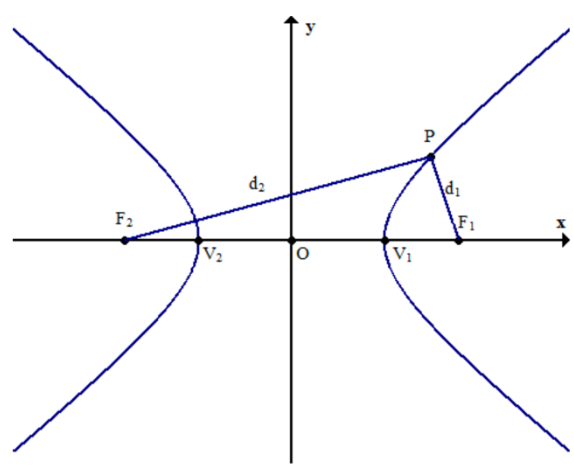
Locus Definition of a Hyperbola

A Hyperbola is the set of all coplanar points where the absolute value of the difference of the distances to two fixed points (the foci) is a fixed constant.

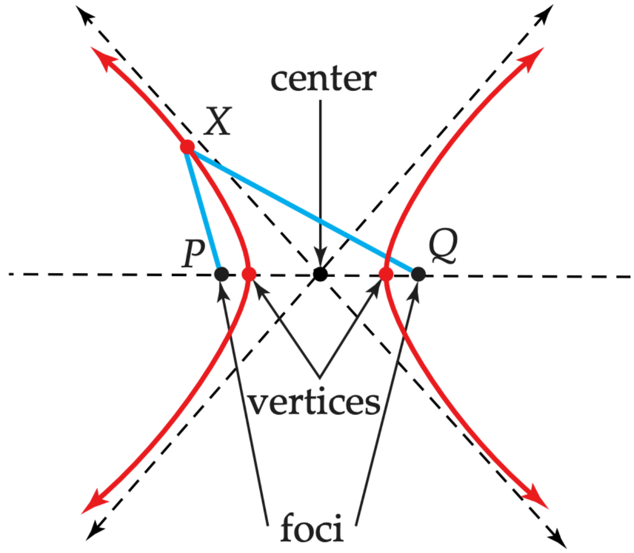
In the horizontal hyperbola at right, we have

$$|d_2 - d_1| = constant = 2a$$

Where a is half the length of the **transverse axis** (or the distance from the **center** to a **vertex**, V.



Here's what the graph of a horizontal Hyperbola looks like with all its parts and vocabulary.



- The **center**, (h, k) is centered between the vertices and foci.
- The **transverse axis** is either a horizontal or vertical axis, and it is the one the Hyperbola crosses (trans-). In this diagram, the transverse axis is horizontal. THE TRANSVERSE AXIS CONTAINS THE VERTICES AND FOCI (similar to the major axis)
- The **conjugate axis** is the other of the horizontal or vertical axis that the graphs does NOT cross. In this diagram, the conjugate axis is vertical.
- The **vertices** are each a units away from the center on the transverse axis. In the diagram above, the vertices are at $(h - a, k)$ & $(h + a, k)$. The length of the transverse axis is $2a$ from vertex to vertices.
- The **foci** are each c units away from the center on the transverse axis. In the diagram above, the foci are at $(h - c, k)$ & $(h + c, k)$.
- The **covertices** are each b units away from the center on the conjugate axis and are "invisible" but essential (as you shall see.)
  In the diagram above, the covertices are at $(h, k + b)$ & $(h, k - b)$.
- The **slant asymptotes** define how wide or narrow the hyperbola is. They are formed by the rectangular box made around the vertices and covertices. In the graph above, the equations of the slant asymptotes (SAs) are $y = k \pm \frac{b}{a}(x - h)$.

Like the ellipse, a Hyperbola has an eccentricity, defined the same way as $e = \frac{c}{a}$. The relation between a, b, and c is similar to the ellipse, but slightly different. For a Hyperbola:

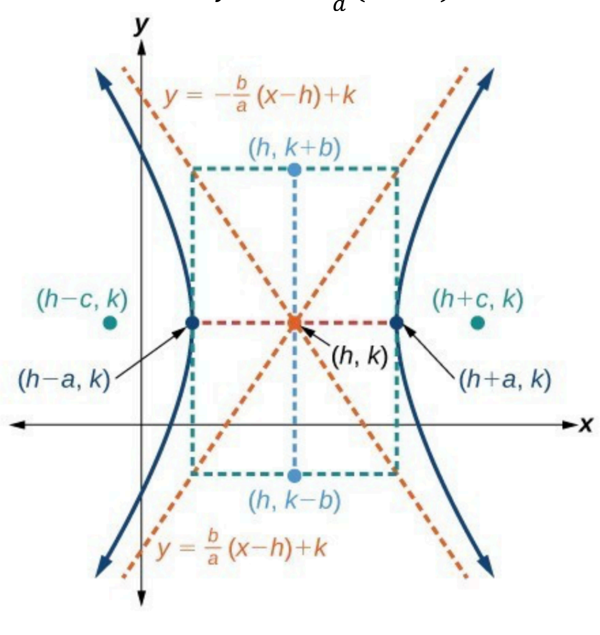
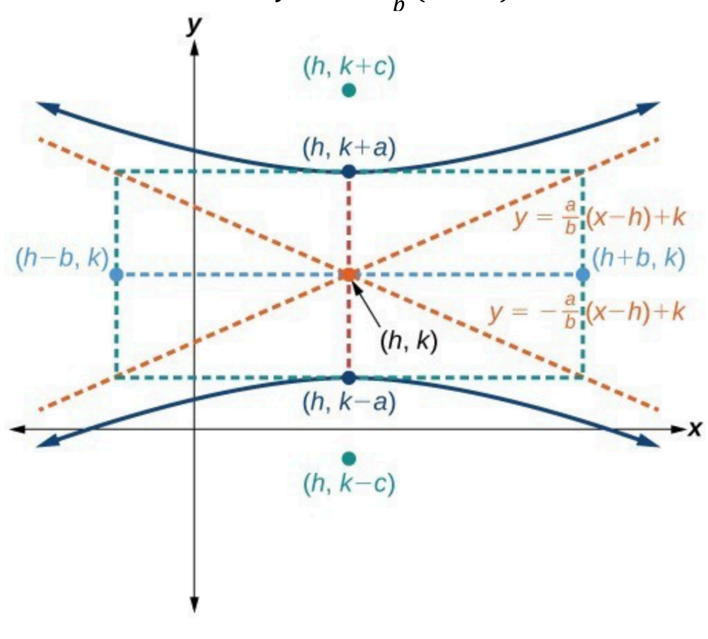
$$a^2 + b^2 = c^2$$

Because the focal length, c, is greater than the vertex length, a, the eccentricity of a Hyperbola is always greater than 1.

$$e > 1$$

Example 1:
Given the general equation $9x^2 - 16y^2 + 36x - 128y - 364 = 0$, explain why this is the equation of a Hyperbola, put the equation into standard form, then sketch the graph finding the foci, eccentricity, domain, range, and equations of the slant asymptotes.

Here is the summary for a Hyperbola in standard form:

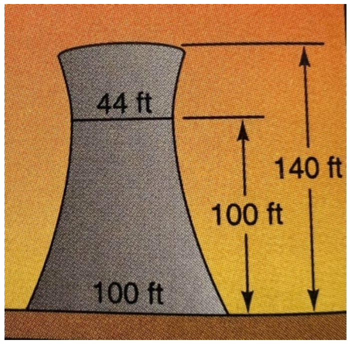
Horizontal Hyperbola	Vertical Hyperbola
*center at origin: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $e = \frac{c}{a} \qquad a^2 + b^2 = c^2$ SAs: $y = k \pm \frac{b}{a}(x-h)$  	*center at origin: $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ $-\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ $e = \frac{c}{a} \qquad a^2 + b^2 = c^2$ SAs: $y = k \pm \frac{a}{b}(x-h)$ 

Example 2:
For the equation $\frac{y^2}{9} - \frac{x^2}{16} = 1$, sketch the graph. Find all the information, including the foci, eccentricity, domain, range, and equations of the SAs.

Example 3:
For the equation $3x^2 - y^2 - 12x - 6y = 0$, sketch the graph. Find all the information, including the foci, eccentricity, domain, range, and equations of the SAs.

Example 4:
Determine the standard equation of the Hyperbola with foci at $(4,1)$ &$(-2,1)$ with a transverse axis of length 2.

Example 5:
Physics The vertical cross section of a cooling tower for a nuclear power plant is a truncated hyperbola. The diameter of the circular base of the tower is 100 ft. The diameter 100 ft above the ground (the narrowest point) is 44 ft. find the equation of the vertical cross section of the cooling tower. Determine the approximate diameter of the top of the tower if is 140 ft high.



*Hyperbolas also have practical reflective properties (anything from one focus passes through the other) that are using in things like LORAN triangulation and Cassegrain telescopes.